We start from the axiom of completeness of $\mathbb{R}$.

Axiom of completeness.

Suppose $A \subset \mathbb{R}$ is nonempty, bounded above. Then, there is $\alpha \in \mathbb{R}$ satisfying both (i),(ii)

(i) $\alpha \geq x \quad \forall x \in A$.

(ii) if $\beta \geq x \quad \forall x \in A$, then $\alpha \leq \beta$.

By trichotomy of $\mathbb{R}$, such α is unique and denoted by $\sup A$, called the supremum of A.

Remark. For a nonempty bounded below set $A \subset \mathbb{R}$, one can define infimum of A similarly. It turns out, $\inf A = -\sup(-A)$.

Monotone convergence theorem is a direct consequence of this axiom.

Monotone Convergence Theorem.

Every monotone increasing, bounded above sequence is convergent.

Proof. Let (x_n) be such a sequence and $A := \{x_n : n \in \mathbb{N}\}$. Let $\alpha := \sup A$. We aim to show that (x_n) converges to α. Let $\epsilon > 0$. Since $\alpha - \epsilon$ cannot be an upper bound, there is $N \in \mathbb{N}$ such that $x_N > \alpha - \epsilon$. Since (x_n) is increasing, $x_n > \alpha - \epsilon$ for all $n > N$. Since α is an upper bound, $\alpha - \epsilon < x_n \leq \alpha$ for all $n > N$. Therefore, $|x_n - \alpha| < \epsilon$ for all $n > N$. □

Remark. For a monotone decreasing, bounded below sequence (x_n), it converges to $-\lim_{n\to\infty} -x_n$, where the limit of $(-x_n)$ is guaranteed by the Monotone convergence theorem.

Digression.

Existence of a monotone subsequence.

Every sequence admits a monotone subsequence.

Proof. Let (x_n) be a sequence. We say that x_n is a peak if $x_k \leq x_n$ for all $k \geq n$. Here we distinguish x_{n_1}, x_{n_2} whenever $n_1 \neq n_2$.

We divide it into two cases. First case: (x_n) has infinitely many peaks. Second case: (x_n) has finitely many peaks.

Case 1:

Let $n_1 := \min\{n \in \mathbb{N} : x_n \text{ is a peak }\}$ and $n_k := \min\{n > n_{k-1} : x_n \text{ is a peak }\}$ for $k \geq 2$. By assumption, $\{n > n_{k-1} : x_n \text{ is a peak }\} \neq \emptyset$ for each $k \geq 2$, hence n_k is well-defined by well-ordering principle of $\mathbb{N}$. Since $n_k > n_{k-1}$ for each $k \geq 2$, (x_{n_k}) is a subsequence of (x_n). It is decreasing.

Case 2:

By assumption, there is $N \in \mathbb{N}$ such that x_n is not a peak whenever $n \geq N$. Let $n_1 := N$ and $n_k := \min\{n > n_{k-1} : x_n > x_{n_{k-1}}\}$ for $k \geq 2$. Since $x_{n_{k-1}}$ is not a peak, $\{n > n_{k-1} : x_n > x_{n_{k-1}}\} \neq \emptyset$ and n_k is well-defined. (x_{n_k}) is a subsequence of (x_n), which is increasing. □

Remark. By Monotone convergence theorem and Existence of a monotone subsequence, every bounded sequence admits a convergent subsequence, which is Bolzano-Weierstrass Theorem.

Nested Interval Theorem.
Suppose (I_k) is a sequence of nondegenerate closed and bounded intervals, such that $I_{k+1} \subset I_k$ for all $k \in \mathbb{N}$. Then,

(i) $\cap_{k=1}^{\infty} I_k \neq \emptyset$

(ii) If $|I_k| \to 0$ as $k \to \infty$, then $\cap_{k=1}^{\infty} I_k = \{\xi\}$ for some $\xi \in \mathbb{R}$.

Proof. Write $I_k = [a_k, b_k]$ with $a_k < b_k$ for each $k \in \mathbb{N}$. Since (a_k) is an increasing sequence bounded by b_1, it converges, say to a. Next, we show that $a \in \cap_{k=1}^{\infty} I_k$. Fix $N \in \mathbb{N}$. Since $a = \sup_{k\in\mathbb{N}} a_k$, $a \geq a_N$. On the other hand, $a_m < b_m \leq b_N$ for every $m \geq N$, therefore, $a = \lim_{k\to\infty} a_k \leq b_N$. These show $a \in I_N$ for any $N \in \mathbb{N}$. That is, $a \in \cap_{k=1}^{\infty} I_k$. This shows (i).
Let $x, y \in \cap_{k=1}^{\infty} I_k$. $|x - y| \leq b_k - a_k = |I_k|$ for every $k \in \mathbb{N}$. Letting $k \to \infty$, $|x - y| = 0$. This shows (ii). □

Remark. Let $a := \lim_{k\to\infty} a_k$, $b := \lim_{k\to\infty} b_k$. Then, $\cap_{k=1}^{\infty} I_k = [a, b]$.

$[0, 1]$ **is uncountable.**

Proof applying Nested interval theorem. Suppose not, let $\{r_1, r_2, ...\}$ be an enumeration of $[0, 1]$. Divide $[0, 1]$ into three closed intervals, each has length $\frac{1}{3}$ and each pair intersects at most one point. Let I_1 be an interval such that $r_1 \notin I_1$. Divide I_1 into three closed intervals, each has length $\frac{1}{3^2}$ and each pair intersects at most one point. Let I_2 be an interval such that $r_2 \notin I_2$. Continuing the process, one admits a sequence of closed intervals (I_k) such that $I_{k+1} \subset I_k$, $r_k \notin I_k$ and $|I_k| = \frac{1}{3^k}$ for each k. By Nested interval theorem (ii), $\cap_{k=1}^{\infty} I_k = \{\xi\}$ for some $\xi \in [0, 1]$. Since $\xi \in I_k$ for each k, $\xi \neq r_k$ for all k and hence $\xi \notin [0, 1]$. Contradiction. There cannot be an enumeration of $[0, 1]$. □

Second proof. Suppose not, let $\{r_1, r_2, ...\}$ be an enumeration of $[0, 1]$. For each $k \in \mathbb{N}$, let $0.a_{k1}a_{k2}a_{k3}...$ be a decimal representation of r_k. A number in $[0, 1]$ admits two decimal representations only if it admits a terminal 0 decimal representation. Let

$$b_k := \begin{cases} 3 & \text{if } a_{kk} \geq 5 \\ 7 & \text{if } a_{kk} < 5 \end{cases}$$

$b := 0.b_1b_2b_3... \in [0, 1]$ admits a unique decimal representation, but for each $k \in \mathbb{N}$, $b_k \neq a_{kk}$. Therefore, $b \neq r_k$ and $b \notin [0, 1]$. Contradiction arises. There cannot be an enumeration of $[0, 1]$. □

Next, we show Bolzano-Weierstrass Theorem from Nested Interval Theorem.
Bolzano-Weierstrass Theorem.
Every bounded sequence admits a convergent subsequence.

Proof. Let (a_n) be a bounded nonconstant sequence. Let $a := \inf_{n\in\mathbb{N}} a_n$ and $b := \sup_{n\in\mathbb{N}} a_n$. Divide $[a,b]$ into two closed intervals with equal length and let I_1 to be one of these two intervals such that $a_n \in I_1$ for infinitely many $n \in \mathbb{N}$. Divide I_1 into two closed intervals with equal length and let I_2 to be one of these two intervals such that $a_n \in I_2$ for infinitely many $n \in \mathbb{N}$. Continuing the process, one admits a sequence of closed intervals (I_k) such that for each $k \in \mathbb{N}$,

(1) $I_{k+1} \subset I_k$

(2) $|I_k| = \frac{b-a}{2^k}$

(3) $a_n \in I_k$ for infinitely many $n \in \mathbb{N}$

By Nested interval theorem (ii), there is $\xi \in \mathbb{R}$ such that $\cap_{k=1}^{\infty} I_k = \{\xi\}$. By (3), one can define $n_1 := \min\{n \in \mathbb{N} : a_n \in I_1\}$ and $n_k := \min\{n > n_{k-1} : a_n \in I_k\}$ for $k \geq 2$. The subsequence (a_{n_k}) converges to ξ. □

Proposition 1: If (a_n) converges to L, then every subsequence (a_{n_k}) converges to L.

Proposition 2: (a_n) converges to L iff every subsequence (a_{n_k}) admits a subsequence $(a_{n_{k_j}})$ converging to L.

Proof of the sufficiency of Proposition 2. Suppose (a_n) does not converge to L. By definition, there is $\epsilon > 0$ such that given any $N \in \mathbb{N}$, $|a_n - L| \geq \epsilon$ for some $n \geq N$. Hence, $n_1 := \min\{n \in \mathbb{N} : |a_n - L| \geq \epsilon\}$ and $n_k := \min\{n > n_{k-1} : |a_n - L| \geq \epsilon\}$ are well-defined. The subsequence (a_{n_k}) satisfying $|a_{n_k} - L| \geq \epsilon$ for each k, admits no subsequence converging to L. Proved by contrapositive. □

Bolzano-Weierstrass can show a generalized nested interval theorem, saying If (F_k) is a sequence of nonempty closed and bounded sets such that $F_{k+1} \subset F_k$ for every $k \in \mathbb{N}$, then

(i) $\cap_{k=1}^{\infty} F_k \neq \emptyset$

(ii) If $\text{diam}(F_k) := \sup\{|x-y| : x, y \in F_k\} \to 0$ as $k \to \infty$, then $\cap_{k=1}^{\infty} F_k = \{\xi\}$ for some $\xi \in \mathbb{R}$.

Here, we adopt the definition that F is said to be closed if given any convergent sequence in F, its limit is also in F.

Proof. Pick $a_k \in F_k$. Since F_1 is a bounded set, by Bolzano-Weierstrass theorem, (a_k) admits a subsequence (a_{n_k}) converging to L. We show that $L \in \cap_{k=1}^{\infty} F_k$. Fix any $N \in \mathbb{N}$, for $k \geq N$, $a_{n_k} \in F_{n_k} \subset F_k \subset F_N$. Since F_N is closed, $L \in F_N$. Hence, $L \in \cap_{k=1}^{\infty} F_k$ and (i) is shown. Proof of (ii) is similar to the proof of nested interval theorem (ii). □

An important application of Bolzano-Weierstrass theorem is to show the Cauchy criterion.

Cauchy Criteria.
(a_n) is convergent iff (a_n) is Cauchy.

Definition. (a_n) is said to be Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for every $n, m \geq N$.
Equivalently, for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $|a_{n+p} - a_n| < \epsilon$ for every $n \geq N$ and $p \in \mathbb{N}$.
That is, $\lim_{n\to\infty} sup_{p\in\mathbb{N}} |a_{n+p} - a_n| = 0$.

Proof of sufficiency of Cauchy criteria. Let (a_n) be a Cauchy sequence. We show the following

(i) (a_n) is bounded

(ii) (a_n) admits a convergent subsequence

(iii) If a Cauchy sequence admits a convergent subsequence, then it converges to its subsequential limit.

By definition of Cauchy, there is $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for every $n, m \geq N$. Therefore, $|a_n| \leq \max\{|a_1|, |a_2|, ..., |a_{N-1}|, |a_N| + 1\}$ for every $n \in \mathbb{N}$ and this shows (i). (ii) follows from (i) and Bolzano-Weierstrass theorem. For (iii),

Suppose (a_{n_k}) is a subsequence of (a_n), converging to L. Let $\epsilon > 0$.

(a) There is $N \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2}$ for every $n, m \geq N$.

(b) There is $K \in \mathbb{N}$ such that $|a_{n_k} - L| < \frac{\epsilon}{2}$ for every $k \geq K$.

Let $p := \max\{N, K\}$. Since $n_p \geq p \geq N$, from (a), we have $|a_n - a_{n_p}| < \frac{\epsilon}{2}$ for every $n \geq N$.
Since $p \geq K$, from (b), we have $|a_{n_p} - L| < \frac{\epsilon}{2}$. By triangle inequality, $|a_n - L| < \epsilon$ for every $n \geq N$. This shows (iii) and the theorem follows. □